Bosonic and fermionic Weinberg-Joos $(j,0)\oplus(0,j)$ states of arbitrary spins as Lorentz-tensors or tensor-spinors and second order theory

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Abstract

We propose a general method for the description of arbitrary single spin-j states transforming according to $(j,0)\oplus(0,j)$ carrier spaces of the Lorentz algebra in terms of Lorentz-tensors for bosons, and tensor-spinors for fermions, and by means of second order Lagrangians. The method allows to avoid the cumbersome matrix calculus and higher ∂^{2j} order wave equations inherent to the Weinberg-Joos approach. We start with reducible Lorentz-tensor (tensor-spinor) representation spaces hosting one sole $(j,0)\oplus(0,j)$ irreducible sector and design there a representation reduction algorithm based on one of the Casimir invariants of the Lorentz algebra. This algorithm allows us to separate neatly the pure spin-j sector of interest from the rest, while preserving the separate Lorentz– and Dirac indexes. However, the Lorentz invariants are momentum independent and do not provide wave equations. Genuine wave equations are obtained by conditioning the Lorentz-tensors under consideration to satisfy the Klein-Gordon equation. In so doing, one always ends up with wave equations and associated Lagrangians that are second order in the momenta. Specifically, a spin-3/2 particle transforming as $(3/2,0)\oplus(0,3/2)$ is comfortably described by a second order Lagrangian in the basis of the totally antisymmetric Lorentz tensor-spinor of second rank, $\Psi_{[\mu\nu]}$. Moreover, the particle is shown to propagate causally within an electromagnetic background. In our study of $(3/2,0)\oplus(0,3/2)$ as part of $\Psi_{[\mu\nu]}$ we reproduce the electromagnetic multipole moments known from the Weinberg-Joos theory. We also find a Compton differential cross section that satisfies unitarity in forward direction. The suggested tensor calculus presents

itself very computer friendly with respect to the symbolic software FeynCalc.

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1 Introduction

Particles of high-spins [1] continue being among the most enigmatic challenges in contemporary theoretical physics. The difficulties in their descriptions, both at the classical-, and the quantum-field theoretical levels, are well known and take their origin from the circumstance that such particles are most naturally described by differential equations of orders twice their respective spins [2]-[5]. Higher-order theories are difficult to tackle and various strategies have been elaborated over the years to lower the order of the corresponding differential equations, the linear ones by Rarita-Schwinger [7] being the most popular so far. However, the latter framework is plagued by various inconsistencies, the acausal propagation within an electromagnetic environment [8], the violation of unitarity in Compton scattering in the ultraviolet in schemes with minimal gauge couplings [9], and the violation of Lorentz-symmetry upon quantization, being the most serious ones. In parallel, also second order spin-$\frac{1}{2}$ [10], [11], [12], [13] and spin-$\frac{3}{2}$ [14] fermion theories have been developed by different authors and shown to provide a reasonable compromise between the rigorous linear– and the natural higher-order descriptions in so far as they were able to circumvent, among others, the acausality problem and the related violation of unitarity in Compton scattering [15]. However, for spins higher than $\frac{3}{2}$ no second order theory has been developed so far. It is the goal of the present work to fill this gap. The interest in such a study is motivated by the observation that distinct representation spaces of the Lorentz algebra $so(1,3)$ describe particles of different physical properties. For example, due to the representation dependence of the boost operator, the electromagnetic quadrupole and octupole moments of fundamental particles with spin-$\frac{3}{2}$ transforming in the four-vector spinor come out different from those of particles transforming as $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ [16].

Same holds valid regarding spin-1 in the four-vector, $\left(\frac{1}{2},\frac{1}{2}\right)$, versus the antisymmetric tensor, $(1,0)\oplus(0,1)$ [17]. In view of the expected production of new particles in the experiments run by the Large Hadron Collider it is important to have at ones disposal a reliable and comfortable to deal with universal calculation scheme for high spins transforming in carrier spaces of the Lorentz algebra different from the totally symmetric tensors of common use, the Weinberg-Joos states being the prime candidates. The present study is devoted to the elaboration of such a scheme.

The path we take is to embed single spin-j Weinberg-Joos states [2]-[5], $(j,0)\oplus(0,j)$, into direct sums of properly selected irreducible $so(1,3)$ representation spaces which join to reducible representation spaces that are large enough as to allow to be equipped by Lorentz, and if needed, separate Dirac indexes. Then we pin down the state of our interest in a two step procedure. First we pin down the $(j,0)\oplus(0,j)$ irreducible representation space by means of a momentum independent (static) projector designed on the basis of one of the Casimir invariants of the Lorentz algebra and then impose on it the Klein-Gordon equation. In this fashion, a second order formalism for any single-spin valued Weinberg- Joos state is furnished. The scheme also allows for an extension to spin-j as the highest spins in the two-spin valued representation spaces, $\left(j-\frac{1}{2},\frac{1}{2}\right)\oplus\left(\frac{1}{2},j-\frac{1}{2}\right)$.

The paper is organized as follows. In the next section we formulate the concepts underlying our suggested method. In section III we design the calculation algorithm and write down a second order master equation for any single spin-j transforming as $(j,0)\oplus(0,j)$. In same section we apply the scheme elaborated to the description of the particular case of spin-$\frac{3}{2}$. In section IV we present the spin-$\frac{3}{2}$ Lagrangian and couple it to the electromagnetic field, find the electromagnetic current, and calculate the associated electromagnetic multipole moments. We show that the observables obtained in this fashion reproduce the predictions reported in [16] where the pure spin-$\frac{3}{2}$ states have been considered in the standard way as eight-dimensional spinors. Also there we show that the pure spin-$\frac{3}{2}$ sector of the antisymmetric tensor spinor describes a particle that propagates causally within an electromagnetic environment. Finally, we calculate the process of Compton scattering off $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ and report on a finite differential cross section in forward direction. The paper closes with brief conclusions and has one Appendix.

2 Lorentz tensors and tensor-spinors for bosons and fermions of any spin and second order wave equations

The method for high-spin description advocated in this work is based upon representation spaces of the Lorentz algebra $so(1,3)$, which are as a rule different from those of common use. While the representation spaces underlying the Weinberg-Joos formalism are of non-tensorial nature, those underlying the Rarita-Schwinger framework are totally symmetric tensors. Our method permits to consider besides totally symmetric also totally antisymmetric Lorentz tensors, and also such of mixed symmetries. Our idea is to embed $(j,0)\oplus(0,j)$ carrier spaces of $so(1,3)$ into finite direct sums of properly chosen dummy irreducible representation spaces with the aim to end up with reducible representation spaces of minimal dimensionality which are nonetheless large enough as to allow to be equipped by Lorentz– (and if needed, separate Dirac) indexes, i.e.

$$\Psi_{\mu_1,..\mu_t} \simeq [(j,0)\oplus(0,j)]\oplus\Sigma_{(k,l)}\, n_{(kl)}\, [(j_k,j_l)\oplus(j_l,j_k)]\,. \tag{1}$$

For example, pure spin-$\frac{3}{2}$ can be embedded into the totally antisymmetric tensor of second rank, $B_{[\mu\nu]}$, with Dirac spinor components, ψ, a representation space that is reducible according to,

$$\begin{aligned}
\Psi_{[\mu\nu]} &\simeq B_{[\mu\nu]}\otimes\psi\\
&\simeq [(1,0)\oplus(0,1)]\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\\
&\longrightarrow \left[\left(\mathbf{\frac{3}{2}},\mathbf{0}\right)\oplus\left(\mathbf{0},\mathbf{\frac{3}{2}}\right)\right]\oplus\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\\
&\oplus \left[\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right)\right]\\
&\longrightarrow \left[\left(\mathbf{\frac{3}{2}},\mathbf{0}\right)\oplus\left(\mathbf{0},\mathbf{\frac{3}{2}}\right)\right]\oplus\psi_\mu,\\
\psi_\mu &\simeq \left(\frac{1}{2},\frac{1}{2}\right)\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right],
\end{aligned} \tag{2}$$

where we use $[\mu\nu]$ to denote an antisymmetric pair of indexes. Spin-2 is part

of the antisymmetric tensor with anti-symmetric tensor components,

$$\begin{aligned}\Phi_{[\mu\nu][\eta\gamma]} &\simeq B_{[\mu\nu]}\otimes B_{[\eta\gamma]}\\ &= [(1,0)\oplus(0,1)]\otimes[(1,0)\oplus(0,1)]\\ &\longrightarrow [(\mathbf{2},0)\oplus(0,\mathbf{2})]\oplus 2\,(0,0)\oplus 2\,(1,1)\\ &\oplus [(1,0)\oplus(0,1)]\,,\end{aligned}\tag{3}$$

where the numbers in front of the irreps indicate their multiplicity upon reduction.

Similarly, spin-$\frac{5}{2}$ can be viewed as a resident of the direct product of the antisymmetric tensor-tensor from (3) with a Dirac spinor, giving rise to $\Psi_{[\mu\nu][\eta\gamma]}$, a representation space reducible according to

$$\begin{aligned}\Phi_{[\mu\nu][\eta\gamma]} &\otimes \psi\simeq\Psi_{[\mu\nu][\eta\gamma]}\\ &\simeq [(1,0)\oplus(0,1)]\otimes[(1,0)\oplus(0,1)]\\ &\otimes \left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\\ &\longrightarrow \left[\left(\frac{\mathbf{5}}{\mathbf{2}},0\right)\oplus\left(0,\frac{\mathbf{5}}{\mathbf{2}}\right)\right]\oplus 3\left[\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right)\right]\\ &\oplus 2\left[\left(1,\frac{3}{2}\right)\oplus\left(\frac{3}{2},1\right)\right]\oplus 3\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]\\ &\oplus 2\left[\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)\right]\oplus\left[\left(2,\frac{1}{2}\right)\oplus\left(\frac{1}{2},2\right)\right].\end{aligned}\tag{4}$$

In order to pin down the $(j,0)\oplus((0,j)$ sector of our interest we have to remove the dummy irreducible sectors in the above reducible tensorial representation spaces. For this purpose we develop an algorithm on the basis of static projectors constructed from one of the Casimir invariants of the Lorentz algebra. Such projectors have the property to unambiguously identify and exclude anyone of the irreducible representation spaces, no matter whether single- or multiple-spin valued [19]. Below we outline this representation reduction algorithm.

2.1 A Casimir invariant of the Lorentz algebra and the $(j,0)\oplus(0,j)$ tracking algorithm

The Lorentz algebra has a Casimir operator, denoted by F and given in [19] in terms of the Lorentz-group generators, $M_{\mu\nu}$, as

$$[F]_{AB} \;=\; \frac{1}{4}[M^{\mu\nu}]_A{}^C[M_{\mu\nu}]_{CB}, \tag{5}$$

with the capital Latin letters A,B, C, ... standing for the generic indexes characterizing the representation space of interest. Its eigenvalue problem for any irreducible representation spaces of the type $(j_1,j_2)\oplus(j_2,j_1)$, with the generic representation functions denoted by, $w^{(j_1,j_2)}$, reads,

$$\begin{aligned} F\,w^{(j_1,j_2)} &= c_{(j_1,j_2)}w^{(j_1,j_2)}, \\ c_{(j_1,j_2)} &= j_1(j_1+1)+j_2(j_2+1)=\frac{1}{2}\left(K(K+2)+M^2\right) \end{aligned} \tag{6}$$

where

$$K=j_1+j_2, \quad M=|j_1-j_2|. \tag{7}$$

On the basis of F we design the following momentum independent Lorentz projector, $\mathcal{P}_F^{(j_1,j_2)}$,

$$\mathcal{P}_F^{(j_1,j_2)}w^{(j_1,j_2)} \;=\; \left[\Pi_{kl}\times\left(\frac{F-c_{(j_k,j_l)}}{c_{(j_1,j_2)}-c_{(j_k,j_l)}}\right)\right]w^{(j_k,j_l)}=w^{(j_1,j_2)}, \tag{8}$$

where $\Pi_{kl}\times$ denotes the operation of successive multiplication, $c_{(j_1,j_2)}$ is the F eigenvalue of the searched sector, $(j_1,j_2)\oplus(j_2,j_1)$, while $c_{(j_k,j_l)}$ are the F eigenvalues of the dummy sectors, $(j_k,j_l)\oplus(j_l,j_k)$, of the hosting tensor, which need all to be excluded. The mayor advantage of such projectors is that they are of zeroth order in the momenta, and do not contribute at all to the order of the wave equations. In what follows we shall mainly consider only such reducible Lorentz tensors (or, tensors-spinors) which allow the spin-j of our interest to reside within one single-spin valued irreducible subspace, $(j,0)\oplus(0,j)$, i.e. in

$$(j_1,j_2)\oplus(j_2,j_1): \quad j=j_1, \quad j_2=0. \tag{9}$$

However, in the next section we show that the algorithm suggested allows for an extension also toward double-spin valued spaces such as $j, j' \in (j_1, j_2) \oplus (j_2, j_1)$ with $j_2 = \frac{1}{2}$, i.e.

$$\left(j_1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, j_1\right): \quad \left\{ \begin{array}{l} j = j_1 - \frac{1}{2}, \\ j' = j_1 + \frac{1}{2}. \end{array} \right. \tag{10}$$

2.2 Second order master equations for any pure spin

The dynamics into the spin-j sectors of interest from above is introduced by implementing the mass-shell condition,

$$\frac{P^2}{m^2} w^{(j,0)} \quad = \quad w^{(j,0)}, \tag{11}$$

for the case of single spins transforming as $(j, 0) \oplus (0, j)$. For double-spin valued representation spaces, the Lorentz projector only tracks down the $\left(j_1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, j_1\right)$ sector as a whole, but does not distinguish between its $j = \left(j_1 - \frac{1}{2}\right)$, and $j' = \left(j_1 + \frac{1}{2}\right)$ residents. In order to single out, say, the spin-j, a different projector, here denoted by $\mathcal{P}^{(m,j)}_{\mathcal{W}^2}(p)$, and based on the squared Pauli-Lubanski vector, has to be employed. The mass projector, P^2/m^2, and the spin-j projector, $\mathcal{P}^{(m,j)}_{\mathcal{W}^2}(p)$, will be occasionally referred to as "Poincaré projectors" [14] in reference to the fact that they express in terms of the Casimir invariants of the Poincaré algebra, the squared four momentum, P^2, and the squared Pauli-Lubanski vector, $\mathcal{W}^2(p)$, as

$$\mathcal{P}^{(m,j)}_{\mathcal{W}^2}(p)\, w^{\left(j_1, \frac{1}{2}\right);j} \quad = \quad w^{\left(j_1, \frac{1}{2}\right);j}, \tag{12}$$

$$\mathcal{P}^{(m,j)}_{\mathcal{W}^2}(p) \quad = \quad \frac{P^2}{m^2} \frac{\mathcal{W}^2(p) - \epsilon_{j'}}{\epsilon_j - \epsilon_{j'}}. \tag{13}$$

Here, $\mathcal{W}^\mu(p)$ denotes the Pauli-Lubanski (pseudo)vector, defined as

$$\left(\mathcal{W}^\mu\right)_{AB}(p) = \frac{1}{2} \epsilon_{\lambda\rho\sigma\mu} \left(M^{\rho\sigma}\right)_{AB} p^\mu, \tag{14}$$

where $M^{\rho\sigma}$ are the generators of the Lorentz algebra in the representation space of interest, while A, and B are again as already mentioned above the sets of indexes that characterize the dimensionality of that very representation space. Furthermore, $\epsilon_j = -p^2 j(j+1)$, and $\epsilon_{j'} = -p^2 j'(j'+1)$, are the

respective eigenvalues of the eigenstates of the operator $\mathcal{W}^2(p)$, corresponding to the spins-j, and j', residing within $\left(j_1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, j_1\right)$, with the common mass-$m$ satisfying $m^2 = p^2$.

Combining now the equation (8) either with (11), or with (12)–(14), the following master equations emerge,

$$\text{for} \quad j \in (j,0) \oplus (0,j): \qquad \left[\frac{P^2}{m^2}\mathcal{P}_F^{(j,0)}\right]_B{}^A w_A^{(j,0)} = w_B^{(j,0)}, \tag{15}$$

and

$$\text{for } j \in \left(j_1, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, j_1\right): \qquad \left[\frac{P^2}{m^2}\mathcal{P}_{\mathcal{W}^2}^{(m,j)}(p)\mathcal{P}_F^{\left(j_1,\frac{1}{2}\right)}\right]_D{}^C w_C^{\left(j_1,\frac{1}{2}\right);j} = w_D^{\left(j_1,\frac{1}{2}\right);j}, \tag{16}$$

respectively. Here, the functions, $w^{(j,0)}$, and $w^{\left(j_1,\frac{1}{2}\right);j}$, have the property each to simultaneously diagonalize both the Lorentz and the Poincaré projectors. The indexes A for bosons are given by, $A = \mu_1, \mu_2 ... \mu_t$, while those for fermions carry in addition a Dirac index, denoted by small Latin letters, a, b, etc. according to $A = \mu_1\mu_2...\mu_t; a$. Along this path, one necessarily encounters Lagrangians that are second order in the momenta.
Second order fermion approaches have traditions in field theory [10],[11], and are of growing popularity in QED as well as in QCD [20], [21], [18].
The present work focuses on the description of the pure spin-$\frac{3}{2}$ Weinberg-Joos state, $\left(\frac{3}{2}, 0\right) \oplus \left(0, \frac{3}{2}\right)$ as part of the antisymmetric tensor-spinor of second rank in (2).

Fermionic representation spaces of the types (2), (4) have been earlier employed by Niederle and Nikitin in [22] in a linear framework of the Rarita-Schwinger type, with the special emphasis on spin-$\frac{3}{2}$. However, the focus of [22] has been in first place the separation of parities, while the question on the precise assignment of spin-$\frac{3}{2}$ to the irreducible $\left(\frac{3}{2}, 0\right) \oplus \left(0, \frac{3}{2}\right)$ sector of the anti-symmetric tensor spinor of second rank has been left aside. Moreover, differently from the present work, no physical observables have been calculated in [22] for the sake of a comparison to predictions by other formalisms. Finally, the scheme of [22] confines to fractional spins alone, while the one elaborated here applies to both fermions and bosons.

3 Pure spin-$\frac{3}{2}$ in $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ as part of the anti-symmetric Lorentz tensor-spinor of second rank

3.1 The anti-symmetric Lorentz tensor spinor of second rank

The antisymmetric Lorentz tensor-spinor of second rank has been defined in eq. (2) and is the ordinary anti-symmetric Lorentz-tensor of second rank , $(1,0)\oplus(0,1)$, with Dirac spinor, $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$, components. The $(1,0)\oplus(0,1)$ sector is well known and has been frequently elaborated in the literature, listed among others in [23],[16]. We here present it in the momentum space, and denote it by, $\mathcal{B}^{[\alpha\beta]}(\mathbf{p},\lambda')$ with $\mathbf{p}$ standing for the three momentum, and λ' denoting the polarization quantum number, taking the values $\lambda'=\pm1,0$. The spin-1 tensor $\mathcal{B}^{[\alpha\beta]}(\mathbf{p},\lambda')$ allows for a representation in terms of the three spin-1 basis states, $\eta^\alpha(\mathbf{p},1,\lambda')$, and the one spin-0 state, $\eta^\alpha(\mathbf{p},0,0)$, spanning the four-vector space, $\left(\frac{1}{2},\frac{1}{2}\right)$. These states have been constructed for example in [24], [16] and are summarized in the equations (123)–(126) in the Appendix below. In terms of the aforementioned $\left(\frac{1}{2},\frac{1}{2}\right)$ basis vectors, the tensor under discussion expresses as,

$$\mathcal{B}^{[\alpha\beta]}(\mathbf{p},\lambda')=\eta^\alpha(\mathbf{p},0,0)\eta^\beta(\mathbf{p},1,\lambda')-\eta^\alpha(\mathbf{p},1,\lambda')\eta^\beta(\mathbf{p},0,0). \tag{17}$$

Now the tensor-spinor of interest here, to be denoted by $\mathcal{T}_\pm^{[\alpha\beta]}(\mathbf{p},\lambda',\lambda'')$, its dual being $\widetilde{\mathcal{T}}_\pm^{[\alpha\beta]}(\mathbf{p},\lambda',\lambda'')$, reads,

$$\mathcal{T}_\pm^{[\alpha\beta]}(\mathbf{p},\lambda',\lambda'') \;=\; \mathcal{B}^{[\alpha\beta]}(\mathbf{p},\lambda')\otimes u_\pm(\mathbf{p},\lambda''), \tag{18}$$

where $u_+(\mathbf{p},\lambda'')$ and $u_-(\mathbf{p},\lambda'')$ denote in their turn the $u(\mathbf{p},\lambda'')$ and the $v(\mathbf{p},\lambda'')$ Dirac spinors of positive/negative parities, and $\lambda''=\pm\frac{1}{2}$. This tensor is reducible according to (2) and its irreducible sectors can be identified by means of projectors constructed from one of the Casimir invariants of the Lorentz algebra, an issue on which we shed light in subsection 3.3 below. Before that, in the subsequent section we present the generators of the Lorentz algebra in the anti-symmetric tensor-spinor of second rank.

3.2 The Lorentz algebra generators in the anti-symmetric tensor spinor

The generators within the anti-symmetric tensor-spinor (ATS) of second rank are

$$
\begin{aligned}
[M^{ATS}_{\mu\nu}]_{[\alpha\beta][\gamma\delta]} &= [M^{AT}_{\mu\nu}]_{[\alpha\beta][\gamma\delta]}\mathbf{1}^S + \mathbf{1}_{[\alpha\beta][\gamma\delta]}\ \left[M^S_{\mu\nu}\right], && (19)\\
\mathbf{1}_{[\alpha\beta][\gamma\delta]} &= \frac{1}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), && (20)\\
M^S_{\mu\nu} &= \frac{1}{2}\boldsymbol{\sigma}_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. && (21)
\end{aligned}
$$

Here, $[M^{AT}_{\mu\nu}]_{[\alpha\beta][\gamma\delta]}$ are the generators within the anti-symmetric tensor (AT) space, $\mathbf{1}_{[\alpha\beta][\gamma\delta]}$ stands for the identity in this space, while $\mathbf{1}^S$ and $M^S_{\mu\nu}$ are the unit operator and the generators within the Dirac space (21), where γ_μ are the standard Dirac matrices. In what follows we shall suppress the Dirac indexes for the sake of avoiding cumbersome notations and will keep only the Lorentz indexes. We always will mark anti-symmetric pairs of Lorentz indexes by [...] any times when more than one pair is involved.

The $[M^{AT}_{\mu\nu}]_{[\alpha\beta][\gamma\delta]}$ generators express in terms of the generators in the four-vector,

$$
[M^V_{\mu\nu}]_{\alpha\beta} = i(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}), \qquad (22)
$$

as,

$$
\begin{aligned}
\left[M^{AT}_{\mu\nu}\right]_{[\alpha\beta][\gamma\delta]} &= \frac{1}{2}\Big(\left[M^V_{\mu\nu}\right]_{\alpha\gamma} g_{\beta\delta} + g_{\alpha\gamma}\left[M^V_{\mu\nu}\right]_{\beta\delta} - \left[M^V_{\mu\nu}\right]_{\alpha\delta} g_{\beta\gamma} - g_{\alpha\delta}\left[M^V_{\mu\nu}\right]_{\beta\gamma}\Big)\\
&= -2\ \mathbf{1}_{[\alpha\beta]}{}^{[\kappa\sigma]}\left[M^V_{\mu\nu}\right]_\sigma{}^\rho \mathbf{1}_{[\rho\kappa][\gamma\delta]}. \qquad (23)
\end{aligned}
$$

Then, the explicit expression for the Casimir invariant F in (6) takes the form,

$$
[F]_{[\alpha\beta][\gamma\delta]} = -\frac{1}{8}\Big(\boldsymbol{\sigma}_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\delta} - \boldsymbol{\sigma}_{\gamma\delta}\boldsymbol{\sigma}_{\alpha\beta} - 22\,\mathbf{1}_{[\alpha\beta][\gamma\delta]}\Big). \qquad (24)
$$

3.3 The Lorentz projector on the irreducible $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ sector

In the antisymmetric tensor-spinor space under investigation there are three Lorentz sectors of the type $(j_2, j_1)\oplus(j_1, j_2)$, corresponding to $(j_2, j_1) = \left(\frac{1}{2}, 0\right)$,

$\left(\frac{1}{2},1\right)$, and $\left(\frac{3}{2},0\right)$. The associated representation functions, $w^{(j_1,j_2)}$, are characterized by their $c_{(j_1,j_2)}$ eigenvalues with respect to the F invariant according to the equations (6), and (7),

$$Fw^{(j_2,j_1)} = c_{(j_1,j_2)}w^{(j_2,j_1)} = \frac{1}{2}(K(K+2)+M^2)w^{(j_1,j_2)}, \tag{25}$$

with

$$K = j_1 + j_2, \qquad M = |j_1 - j_2|. \tag{26}$$

All three eigenvalues are different and given by,

$$c_{\left(\frac{1}{2},0\right)} = \frac{3}{4}, \qquad c_{\left(\frac{1}{2},1\right)} = \frac{11}{4}, \qquad c_{\left(\frac{3}{2},0\right)} = \frac{15}{4}. \tag{27}$$

This allows us to define the three independent operators, $\mathcal{Q}^{\left(\frac{1}{2},0\right)}$, and $\mathcal{Q}^{\left(\frac{1}{2},1\right)}$ as

$$\mathcal{Q}^{(j_1,j_2)} \quad = \quad F - c_{(j_1,j_2)}\mathbf{1}, \tag{28}$$

which have the properties to remove from the antisymmetric tensor-spinor in (2) the dummy $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$, and $\left(\frac{1}{2},1\right)\oplus\left(\frac{1}{2},1\right)$, companions to $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$. Specifically, the Lorentz projector for the irreducible sector $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ is cast into the following form,

$$\mathcal{P}_F^{\left(\frac{3}{2},0\right)} = \frac{\mathcal{Q}^{\left(\frac{1}{2},1\right)}\mathcal{Q}^{\left(\frac{1}{2},0\right)}}{\left(c_{\left(\frac{3}{2},0\right)} - c_{\left(\frac{1}{2},1\right)}\right)\left(c_{\left(\frac{3}{2},0\right)} - c_{\left(\frac{1}{2},0\right)}\right)}, \tag{29}$$

with $c_{\left(\frac{3}{2},0\right)}$, $c_{\left(\frac{1}{2},1\right)}$, $c_{\left(\frac{1}{2},0\right)}$ from eq. (27). In this way we find the following Lorentz projector on $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$,

$$\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta][\gamma\delta]} \quad = \quad \frac{1}{8}(\boldsymbol{\sigma}_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\delta} + \boldsymbol{\sigma}_{\gamma\delta}\boldsymbol{\sigma}_{\alpha\beta}) - \frac{1}{12}\boldsymbol{\sigma}_{\alpha\beta}\boldsymbol{\sigma}_{\gamma\delta}, \tag{30}$$

which satisfies the conditions,

$$\gamma^\alpha \left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta][\mu\nu]} \quad = \quad 0, \tag{31}$$

$$\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta][\mu\nu]} \gamma^\mu \quad = \quad 0. \tag{32}$$

3.4 The spin-$\frac{3}{2}$ wave equation

We now consider the action of the Lorentz projector in (29) on the tensor-spinor and its dual and generate in this way the set of twenty four momentum space wave functions, $\left[\phi_{\pm}^{(\frac{3}{2},0)}\left(\mathbf{p},\frac{3}{2},\lambda',\lambda''\right)\right]^{[\alpha\beta]}$, and $\left[\widetilde{\phi}_{\pm}^{(\frac{3}{2},0)}\left(\mathbf{p},\frac{3}{2},\lambda',\lambda''\right)\right]^{[\alpha\beta]}$, as,

$$
\begin{aligned}
\left[\phi_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda',\lambda''\right)\right]^{[\alpha\beta]} &= \left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]}\mathcal{T}_{\pm}^{[\gamma\delta]}(\mathbf{p},\lambda',\lambda''),\\
\left[\widetilde{\phi}_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda',\lambda''\right)\right]^{[\alpha\beta]} &= \left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]}\widetilde{\mathcal{T}}_{\pm}^{[\gamma\delta]}(\mathbf{p},\lambda',\lambda''),\\
\lambda' &= \pm 1, 0,\ \lambda'' = \pm\frac{1}{2},\\
\lambda &= \pm\frac{1}{2},\pm\frac{3}{2}.
\end{aligned}
\tag{33}
$$

After some algebraic manipulations, it can be verified that these are all $\mathcal{P}_F^{\left(\frac{3}{2},0\right)}$ eigenfunctions, meaning that they all reside in $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$, although only eight of them are linearly independent, as it should be. Out of them, a set of pure spin-$\frac{3}{2}$ tensor–spinors of positive and negative parities, henceforth denoted by $\left[w_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]}$, can be constructed and cast into the form of the Lorentz projector $\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\mu\nu]}$ applied to the anti-symmetric combination, $\left[U_{\pm}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\mu\nu]}$, of the four-momentum, p^μ, with a four-vector spinor, $\mathcal{U}_{\pm}^{\nu}\left(\mathbf{p},\frac{3}{2},\lambda\right)$, as explained in eq. (128) in the Appendix. In other words, one can write,

$$
\begin{aligned}
\left[w_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]} &= 2\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\mu\nu]}\left[U_{\pm}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\mu\nu]}\\
&= \frac{2}{m}\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\mu\nu]}p^\mu\mathcal{U}_{\pm}^{\nu}\left(\mathbf{p},\frac{3}{2},\lambda\right),
\end{aligned}
\tag{34}
$$

with $\mathcal{U}_{\pm}^{\nu}\left(\mathbf{p},\frac{3}{2},\lambda\right)$ defined in eqs. (122) in the Appendix. The factor 2 ensures the normalization of these states to (± 1). To the amount $\left[w_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]}$

are considered on their mass shell, they satisfy the Klein-Gordon equation, as explained in (8), (15), and (25). In effect, the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ degrees of freedom are found to solve the following second order equation,

$$\left(\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]}p^2 - m^2\mathbf{1}^{[\alpha\beta]}{}_{[\gamma\delta]}\right)\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\gamma\delta]} = 0. \tag{35}$$

In this fashion, the pure spin-$\frac{3}{2}$ degrees of freedom are manifestly generated in the shape of tensor-spinors. Compared to the eight-component Weinberg-Joos spin-$\frac{3}{2}$ "vectors", listed among others in [16], the eight tensor-spinors, $\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]}$, open an avenue towards efficient tensor calculations of scattering cross-sections off $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ by the help of the symbolic software FeynCalc, a reason for which we consider the equation (33) as the first achievement of the present study, worth reporting. The aforementioned tensor spinors do not satisfy the Dirac equation, due to the nonzero commutator, $\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)},\not p\right]\neq 0$, between the Lorentz-projector and the Feynman slash, $\not p$. This is a crucial circumstance for the neat separation of the Rarita-Schwinger– from the Weinberg-Joos sector in the anti-symmetric Lorentz tensor-spinor of second rank. The tensor-spinors are conditioned through,

$$\gamma^\alpha\gamma^\beta\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]_{[\alpha\beta]} = 0. \tag{36}$$

We now introduce the short-hand $\left[f^{\left(\frac{3}{2},0\right)}(p)\right]^{\alpha\beta\mu}$ for convenience as

$$\left[f^{\left(\frac{3}{2},0\right)}(p)\right]^{\alpha\beta\mu} = \frac{2}{m}\left[\mathcal{P}_F^{(j_1,j_2)}\right]^{[\alpha\beta][\gamma\mu]}p_\gamma, \tag{37}$$

with

$$\left[\overline{f}^{\left(\frac{3}{2},0\right)}(p)\right]^{\alpha\beta}{}_{\mu}\left[f^{\left(\frac{3}{2},0\right)}(p)\right]_{\alpha\beta\nu} = \frac{1}{m^2}\left(g_{\alpha\beta}g_{\mu\nu}-\frac{1}{3}\boldsymbol{\sigma}_{\alpha\mu}\boldsymbol{\sigma}_{\beta\nu}-g_{\alpha\mu}g_{\beta\nu}\right)p^\alpha p^\beta. \tag{38}$$

This allows us to write the tensor-spinors in (34) as

$$\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]} = \left[f^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]^{\alpha\beta}{}_{\mu}\left[\mathcal{U}_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{\mu}. \tag{39}$$

The conjugate tensor-spinors are then introduced as

$$\left[\overline{w}_{\pm}^{(\frac{3}{2},0)}(\mathbf{p},\frac{3}{2},\lambda)\right]_{[\alpha\beta]} = \overline{\mathcal{U}}_{\pm}^{\mu}\left(\mathbf{p},\frac{3}{2},\lambda\right)\left[\overline{f}^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]_{\alpha\beta}{}^{\mu}, \tag{40}$$

were $\left[\overline{f}^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]_{\alpha\beta\mu} = \gamma^0\left(\left[f^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]_{\alpha\beta\mu}\right)^{\dagger}\gamma^0.$

The above spin-$\frac{3}{2}$ tensor-spinors are normalized as

$$\left[\overline{w}_{\pm}^{(\frac{3}{2},0)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]_{[\alpha\beta]}\left[w_{\pm}^{(\frac{3}{2},0)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]} = \pm 1. \tag{41}$$

We now calculate their propagator as,

$$\left[S^{\left(\frac{3}{2},0\right)}(p)\right]_{[\alpha\beta][\gamma\delta]} = \frac{\left[\Delta^{\left(\frac{3}{2},0\right)}(p)\right]_{[\alpha\beta][\gamma\delta]}}{p^2 - m^2 + i\epsilon}, \tag{42}$$

with

$$\left[\Delta^{\left(\frac{3}{2},0\right)}(p)\right]_{[\alpha\beta][\gamma\delta]} = \frac{p^2}{m^2}\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta][\gamma\delta]} + \frac{(m^2-p^2)}{m^2}\mathbf{1}_{[\alpha\beta][\gamma\delta]}. \tag{43}$$

Second order theories present the notorious problem that the propagators are of unspecified spatial parities. For bosons this circumstance does not present an obstacle in so far as bosons and anti-bosons are of equal parities. Such is due to the commutativity of the parity and the charge conjugation operators. A discussion on this issue can be found around the equation (2.34) in the reference [16]. For massive charged fermions, however, for which particles and anti-particles are of opposite parities, the problem is more serious and can affect the quantization procedure. However, we expect that the method would allow for quantization of Majorana fermions (if they were to exist) and massless particles. Moreover, at the level of scattering processes, the problem of parity distinction of charged massive particles still can be attended by constructing the amplitudes for states of a fixed parity and then inserting into the squared amplitudes the relevant parity projector, a path that we take in the evaluation of the process of Compton scattering as presented in subsection 5 of the next section. The method is highlighted in table 1.

Table 1: Glossary of the description in momentum space of pure spin-$\frac{3}{2}$ transforming according to the irreducible $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ sector of the anti-symmetric Lorentz tensor-spinor of second rank, $\left[\mathcal{T}_{\pm}\left(\mathbf{p},\lambda',\lambda''\right)\right]^{[\mu\nu]}$. Here, $\mathcal{U}^{\alpha}_{\pm}\left(\mathbf{p},\frac{3}{2},\lambda\right)$ are the four vector-spinor degrees of freedom residing in the tensor-spinor space and are defined in eqs. (34), (122). The low case ($\pm$) index refers to spinors of either positive, (+), or negative, ($-$), parity, respectively.

$(j,0)\oplus(0,j)$	$\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)\in\left[\mathcal{T}_{\pm}\left(\mathbf{p},\lambda',\lambda''\right)\right]^{[\mu\nu]}$.
Tensor-spinor and its dual:	$\left[\mathcal{T}_{\pm}\left(\mathbf{p},\lambda',\lambda''\right)\right]^{[\mu\nu]}$, eq. (18) $\left[\widetilde{\mathcal{T}}_{\pm}\left(\mathbf{p},\lambda',\lambda''\right)\right]^{[\mu\nu]}$
Lorentz projector:	$[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta][\gamma\delta]}=\frac{1}{8}\Big(\sigma_{\alpha\beta}\sigma_{\gamma\delta}+\sigma_{\gamma\delta}\sigma_{\alpha\beta}\Big)$ - $\frac{1}{12}\sigma_{\alpha\beta}\sigma_{\gamma\delta}$, eq. (30)
Primordial spin-$\frac{3}{2}$ tensor-spinors, and dual tensor-spinors:	$\left[\phi_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda',\lambda''\right)\right]_{[\alpha\beta]}=\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right))}\right]_{[\alpha\beta][\gamma\delta]}$ $\times\left[\mathcal{T}_{\pm}(\mathbf{p},\lambda',\lambda'')\right]^{[\gamma\delta]}$, eqs. (33) $\left[\widetilde{\phi}_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda',\lambda''\right)\right]_{[\alpha\beta]}=\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right))}\right]_{[\alpha\beta][\gamma\delta]}$ $\times\left[\widetilde{\mathcal{T}}_{\pm}(\mathbf{p},\lambda',\lambda'')\right]^{[\gamma\delta]}$
Parity representation:	$\left[w_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]}=\frac{2}{m}\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\mu\nu]}$ $\times p^{\mu}\mathcal{U}^{\nu}_{\pm}\left(\mathbf{p},\frac{3}{2},\lambda\right)$, eq. (34),(128)
Wave equation:	$w_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)_{[\gamma\delta]}=[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta][\gamma\delta]}\frac{p^2}{m^2}$ $\times\left[w_{\pm}^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]}$ eq. (35)
Electromagnetic current:	$j_{\mu}^{\left(\frac{3}{2},0\right)}(\mathbf{p}',\lambda',\mathbf{p},\lambda)=\frac{38}{9}\,e\,\overline{\mathcal{U}}^{\alpha}_{+}\left(\mathbf{p}',\frac{3}{2},\lambda'\right)$ $\Big((p'+p)_{\mu}g_{\alpha\beta}-mg_{\alpha\beta}\gamma_{\mu}-(p'_{\beta}g_{\alpha\mu}+p_{\alpha}g_{\beta\mu})$ $+\frac{20}{38m}(p_{\alpha}p'_{\beta}-p'\cdot p g_{\alpha\beta})\gamma_{\mu}\Big)\mathcal{U}^{\beta}_{+}\left(\mathbf{p},\frac{3}{2},\lambda\right)$, eq. (65)

4 The coupling of the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ sector of the anti-symmetric tensor spinor to the electromagnetic field

As a next step after having designed the appropriate free equations of motion in (15), (35), we introduce the electromagnetic interaction. We here confine to minimal gauging, i.e. the coupling is found by replacing ordinary by covariant derivatives according to,

$$\partial^\mu \longrightarrow D^\mu = \partial^\mu + ieA^\mu, \tag{44}$$

where e is the electric charge of the particle.

4.1 The gauge procedure and the number of spin degrees of freedom upon gauging

In order to obtain the gauged equations, we first pass the states from momentum to position space using the standard quantization prescription, $[w^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)]^{[\alpha\beta]}e^{\mp i\,x\cdot p} \longrightarrow [\psi^{\left(\frac{3}{2},0\right)}(x,\lambda)]^{[\alpha\beta]}$, and then write the momenta in operator form yielding,

$$[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta]}{}^{[\gamma\delta]}\partial^2[\psi^{\left(\frac{3}{2},0\right)}(x,\lambda)]^{[\alpha\beta]} = -m^2[\psi^{\left(\frac{3}{2},0\right)}(x,\lambda)]^{[\gamma\delta]}. \tag{45}$$

We now define at the free particle level a new tensor, $[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}]^{[\gamma\delta]}{}_{[\alpha\beta]}$, as

$$[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta][\gamma\delta]}\partial^\mu\partial^\nu = [\mathcal{P}_F^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta][\gamma\delta]}\partial^2, \tag{46}$$

and then cast the gauged eq. (45) as,

$$[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta][\gamma\delta]}D^\mu D^\nu[\Psi^{\left(\frac{3}{2},0\right)}(x)]^{[\alpha\beta]} = -m^2[\Psi^{\left(\frac{3}{2},0\right)}(x)]^{[\gamma\delta]}, \tag{47}$$

where we denoted by $\left[\Psi^{\left(\frac{3}{2},0\right)}(x)\right]^{[\alpha\beta]}$ the new gauged solutions. In order to guarantee that the gauged solutions continue being eigenstates of the Lorentz projectors and thereby to ensure equality of the number of the degrees of freedom before and after gaugung, the $[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}]_{[\alpha\beta][\gamma\delta]}$ tensor has to satisfy,

$$\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta]}{}^{[\sigma\rho]}\left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]_{[\sigma\rho][\gamma\delta]} = \left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta][\gamma\delta]}. \tag{48}$$

Once the validity of the equation (48) has been ensured, the representations space and therefore the spin after gauging continues being same as before. It should be noticed that the $\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}$ tensor in (46) is determined at the free particle level modulo additive terms leading to vanishing $[\partial^\mu, \partial^\nu]$ commutators prior gauging, which upon gauging give rise to the electromagnetic field strength tensor, $F_{\mu\nu}$ and thereby to non-vanishing contributions to the gauged equations. As first discussed in [14], exploiting this freedom could be of help in achieving causality of particle propagation, and/or unitarity of scattering amplitudes in the ultraviolet.

4.2 Causality of $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ propagation upon gauging

Using the aforementioned freedom we chose $\left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]}$ in such a way that,

$$\begin{aligned}
\left[\Gamma^{\left(\frac{3}{2},0\right)\mu}{}_{\nu}\right]^{[\alpha\beta]}{}_{[\gamma\delta]} &= 4\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta][\sigma\mu]}\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]_{[\sigma\nu][\gamma\delta]} \\
&= \frac{1}{2}\Big(\boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{\sigma}^{\sigma\mu}\boldsymbol{\sigma}_{\sigma\nu}\boldsymbol{\sigma}_{\gamma\delta} + \boldsymbol{\sigma}^{\sigma\mu}\boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{\sigma}_{\gamma\delta}\boldsymbol{\sigma}_{\sigma\nu} \\
&\quad -3\boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{\sigma}^{\sigma\mu}\boldsymbol{\sigma}_{\gamma\delta}\boldsymbol{\sigma}_{\sigma\nu} - 3\boldsymbol{\sigma}^{\sigma\mu}\boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{\sigma}_{\sigma\nu}\boldsymbol{\sigma}_{\gamma\delta}\Big) \\
&\quad +4\boldsymbol{\sigma}^{\sigma\mu}\boldsymbol{\sigma}^{\alpha\beta}\boldsymbol{\sigma}_{\sigma\nu}\boldsymbol{\sigma}_{\gamma\delta},
\end{aligned} \tag{49}$$

is satisfied, thus ensuring validity of (46) and (48). This is the precise tensor that enters the gauged equation for the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ sector in (47).

By virtue of (48), on expects the solutions to (47) to continue behaving as $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ states and depending on only eight degrees of freedom. In order to see this, it is convenient to become aware of the fact that that upon accounting for the Dirac label, the equation (47) in reality stands for a (24×24) dimensional matrix equation. It is straightforward to cross-check that the matrix in (47) in combination with (49) can be block-diagonalized with one of the blocks being precisely the expected (8×8) dimensional one. The corresponding eight-dimensional gauged solutions, $\Psi^{\left(\frac{3}{2},0\right)}(x)$, can be now written as linear combinations within any basis spanning the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$

representation space. In particular, if we choose the orthogonal set of the rest-frame states given by (39), $\Psi^{\left(\frac{3}{2},0\right)}(x)$ can be decomposed according to,

$$\Psi^{\left(\frac{3}{2},0\right)}(x) = \sum_{\lambda''} \left(a_+(x)_{\lambda''} w_+^{\left(\frac{3}{2},0\right)}(\mathbf{0}, \frac{3}{2}, \lambda'') - a_-(x)_{\lambda''} w_-^{\left(\frac{3}{2},0\right)}(\mathbf{0}, \frac{3}{2}, \lambda'') \right). \quad (50)$$

As a reminder, the reduction to eight degrees of freedom is possible by virtue of the condition in (48) imposed by the Lorentz projector on the $\left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]_{[\alpha\beta][\gamma\delta]}$ tensor which amounted to (49). Below we show that the expansion of the gauged solutions in (50) significantly facilitates the proof of their causal propagation within an electromagnetic environment.

The causality and hyperbolicity of the equations of motion of order ≤ 2 in the derivatives are tested using the Courant-Hilbert criterion, which requires us to calculate the so called "characteristic determinant" [8] of the gauged equation. The latter is found by replacing the highest order derivatives by the components of the vectors n^μ, interpreted as the normals to the characteristic surfaces, and which characterize the propagation of the (classical) wave fronts of the gauged equation. If the vanishing of the characteristic determinant demands to have a real-valued time-like component n^0, then the equation is hyperbolic. If this determinant nullifies for $n^\mu n_\mu = 0$, then the equation is in addition causal [8]. The $a_\pm(x)_{\lambda'}$ coefficients are calculated upon substitution of (50) in (47), and making use of the normalization of the basis tensors in (41) as,

$$\left[\overline{w}_\pm^{\left(\frac{3}{2},0\right)}(\mathbf{0}, \frac{3}{2}, \lambda')\right]_{[\alpha\beta]} \left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]} D^\mu D^\nu \left[\Psi^{\left(\frac{3}{2},0\right)}(x)\right]^{[\gamma\delta]} = -m^2 a_\pm(x)_{\lambda'}, \quad (51)$$

meaning that they satisfy the equation,

$$\begin{aligned} \left[\overline{w}_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0}, \frac{3}{2}, \lambda'\right)\right]_{[\alpha\beta]} & \left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]} D^\mu D^\nu \sum_{\lambda''} a_\pm(x)_{\lambda''} \left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0}, \frac{3}{2}, \lambda''\right)\right]^{[\gamma\delta]} \\ &= -m^2 a_\pm(x)_{\lambda'}, \quad (52) \\ D^\mu D^\nu &= \partial^\mu\partial^\nu + ie(\partial^\mu A^\nu) + ieA^\nu\partial^\mu + ieA^\mu\partial^\nu - e^2 A^\mu A^\nu. \quad (53) \end{aligned}$$

The equation (52) represents a system of eight partial differential equation for the eight coefficients $a_\pm(x)_{\lambda''}$, because the polarization label λ'' runs

over the four allowed values, $\lambda'' = \pm\frac{3}{2}, \pm\frac{1}{2}$, and is counted twice because of the two parities (the lower case $\pm$ indexes), of the basis tensor-spinors, $w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda''\right)$. The elements of the characteristic matrix are proportional to n^2, as can been seen from replacing the principal part, $\partial^\mu\partial^\nu$ of $D^\mu D^\nu$ in (53) by $n^\mu n^\mu$ yielding,

$$\begin{aligned}
&\left[\overline{w}_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda'\right)\right]_{[\alpha\beta]}\left[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]}n^\mu n^\nu\\
\times\eta_t&\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda''\right)\right]^{[\gamma\delta]} = \left[\overline{w}_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda'\right)\right]_{[\alpha\beta]}\\
&\times n^2\left[\mathcal{P}_F^{\left(\frac{3}{2},0\right)}\right]^{[\alpha\beta]}{}_{[\gamma\delta]}\eta_t\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda''\right)\right]^{[\gamma\delta]}\\
= n^2&\left[\overline{w}_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda'\right)\right]_{[\alpha\beta]}\eta_t\left[w_\pm^{\left(\frac{3}{2},0\right)}\left(\mathbf{0},\frac{3}{2},\lambda''\right)\right]^{[\alpha\beta]}\\
&= n^2\delta_{\lambda'\lambda''}\delta_{\pm\pm},\\
&t=\pm,\ \eta_+ = -\eta_- = 1.
\end{aligned} \tag{54}$$

The characteristic matrix in question is therefore diagonal and each element equals n^2, correspondingly, its determinant is

$$\mathcal{D}^{\left(\frac{3}{2},0\right)}(n) = (n^2)^8. \tag{55}$$

Nullifying the latter amounts to the condition $n^2 = n^\mu n_\mu = 0$, which guarantees causal propagation within the electromagnetic environment.

4.3 The Lagrangian for $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ as part of the anti-symmetric tensor spinor

The free equations of motion (47) can now be derived from the following Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\text{free}}^{\left(\frac{3}{2},0\right)} &= \left(\partial^\mu[\overline{\psi}^{\left(\frac{3}{2},0\right)}]^A\right)[\Gamma_{\mu\nu}^{\left(\frac{3}{2},0\right)}]_{AB}\partial^\nu[\psi^{\left(\frac{3}{2},0\right)}]^B - m^2[\overline{\psi}^{\left(\frac{3}{2},0\right)}]^A[\psi^{\left(\frac{3}{2},0\right)}]_A,\\
A &= [\mu\nu]\,,\quad B=[\gamma\delta]\,,
\end{aligned} \tag{56}$$

where we suppressed the arguments for the sake of simplifying notations. The gauged Lagrangian then emerges as,

$$\mathcal{L}^{\left(\frac{3}{2},0\right)} = \left(D^{*\,\mu}[\overline{\Psi}^{\left(\frac{3}{2},0\right)}]^A\right)[\Gamma^{\left(\frac{3}{2},0\right)}_{\mu\nu}]_{AB}D^\nu[\Psi^{\left(\frac{3}{2},0\right)}]^B - m^2[\overline{\Psi}^{\left(\frac{3}{2},0\right)}]^A[\Psi^{\left(\frac{3}{2},0\right)}]_A, \tag{57}$$

and its decomposition into free and interacting parts reads,

$$\mathcal{L}^{\left(\frac{3}{2},0\right)} = \mathcal{L}^{\left(\frac{3}{2},0\right)}_{\text{free}} + \mathcal{L}^{\left(\frac{3}{2},0\right)}_{\text{int}}, \tag{58}$$

$$\mathcal{L}^{\left(\frac{3}{2},0\right)}_{\text{int}} = -j^{\left(\frac{3}{2},0\right)}_\mu A^\mu + k^{\left(\frac{3}{2},0\right)}_{\mu\nu} A^\mu A^\nu. \tag{59}$$

Back to momentum space, we extract the following vector and tensor bi-linear forms,

$$j^{\left(\frac{3}{2},0\right)}_\mu(\mathbf{p},\lambda,\mathbf{p}',\lambda') = e\left[\overline{w}^{\left(\frac{3}{2},0\right)}_\pm\left(\mathbf{p}',\frac{3}{2},\lambda'\right)\right]^A[\mathcal{V}^{\left(\frac{3}{2},0\right)}_\mu(p',p)]_{AB}\left[w^{\left(\frac{3}{2},0\right)}_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^B, \tag{60}$$

$$k^{\left(\frac{3}{2},0\right)}_{\mu\nu}(\mathbf{p},\lambda,\mathbf{p}',\lambda') = e\left[\overline{w}^{\left(\frac{3}{2},0\right)}_\pm\left(\mathbf{p}',\frac{3}{2},\lambda'\right)\right]^A[\mathcal{C}^{\left(\frac{3}{2},0\right)}_{\mu\nu}]_{AB}\left[w^{\left(\frac{3}{2},0\right)}_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^B, \tag{61}$$

the first standing for the electromagnetic current. The vertexes are given as,

$$[\mathcal{V}^{\left(\frac{3}{2},0\right)}_\mu(p',p)]_{AB} = [\Gamma^{\left(\frac{3}{2},0\right)}_{\nu\mu}]_{AB}p'^\nu + [\Gamma^{\left(\frac{3}{2},0\right)}_{\mu\nu}]_{AB}p^\nu, \tag{62}$$

$$[\mathcal{C}^{\left(\frac{3}{2},0\right)}_{\mu\nu}]_{AB} = \frac{1}{2}\left([\Gamma^{\left(\frac{3}{2},0\right)}_{\mu\nu}]_{AB} + [\Gamma^{\left(\frac{3}{2},0\right)}_{\nu\mu}]_{AB}\right). \tag{63}$$

The Feynman rules following from this Lagrangian are depicted in Figs. 1, 2, 3. It is not difficult to verify that the one-photon vertex obeys

$$(p'-p)^\mu[\mathcal{V}^{\left(\frac{3}{2},0\right)}_\mu(p',p)]_{AB} = [S^{\left(\frac{3}{2},0\right)}(p')]_{AB} - [S^{\left(\frac{3}{2},0\right)}(p)]_{AB}, \tag{64}$$

which is the Ward-Takahashi identity, as it should be.

4.4 Electromagnetic multipole moments

We now calculate the electromagnetic multipole moments of a particle transforming in the single spin- $\frac{3}{2}$ irreducible Weinberg-Joos sector of the anti-symmetric tensor-spinor of interest from the current in (60), taken between

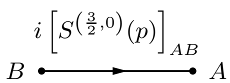

Fig. 1: Feynman rule for the spin-$\frac{3}{2}$ propagator in (42) of a particle transforming in the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ sector of the anti-symmetric Lorentz tensor-spinor of second rank. All the subsequent figures refer to this case.

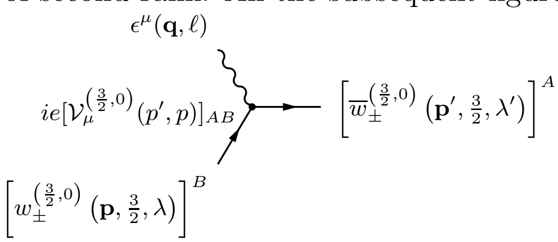

Fig. 2: Feynman rule for the one-photon vertex in (62).

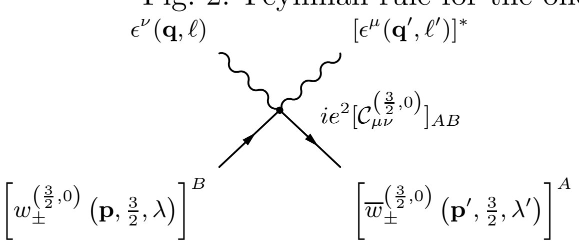

Fig. 3: Feynman rule for the two-photon contact vertex in (63).

positive parities. This current can be further simplified in taking advantage of the equations (34) from above, of eqs. (127)-(128) in the Appendix, and of the mass shell condition, yielding,

$$
\begin{aligned}
j_\mu^{\left(\frac{3}{2},0\right)}(\mathbf{p}',\lambda',\mathbf{p},\lambda) &= \frac{38}{9}\, e\,\overline{\mathcal{U}}^{\alpha}_{+}\Big(\mathbf{p}',\frac{3}{2},\lambda'\Big) \\
&\Big((p'+p)_\mu g_{\alpha\beta} - m g_{\alpha\beta}\gamma_\mu - (p'_\beta g_{\alpha\mu} + p_\alpha g_{\beta\mu}) \\
&+ \frac{20}{38m}(p_\alpha p'_\beta - p'\cdot p\, g_{\alpha\beta})\gamma_\mu\Big)\mathcal{U}^{\beta}_{+}\Big(\mathbf{p},\frac{3}{2},\lambda\Big).
\end{aligned}
\tag{65}
$$

The procedure of extracting the multipole moments from known currents is well established and will not be highlighted here (see for example [25],[16] and references therein for technical details), and amounts to

$$
[Q^0_E(\lambda)] = e, \tag{66}
$$

$$
[Q^1_M(\lambda)] = \frac{2}{3}\left(\frac{e}{2m}\right)\langle M_{12}\rangle, \tag{67}
$$

$$
[Q^2_E(\lambda)] = -\frac{1}{3}\left(\frac{e}{m^2}\right)\langle \mathcal{A}\rangle, \tag{68}
$$

$$
[Q^3_M(\lambda)] = -2\left(\frac{e}{2m^3}\right)\langle \mathcal{B}\rangle, \tag{69}
$$

with

$$
\mathcal{A} = 3M_{12}^2 - \mathbf{J}^2, \qquad \mathcal{B} = M_{12}\left(15M_{12}^2 - \frac{41}{5}\mathbf{J}^2,\right),
$$

$$
\mathbf{J}^2 = M_{12}^2 + M_{13}^2 + M_{23}^2. \tag{70}
$$

The expressions in (66)–(69) fully coincide in form with the corresponding predictions by the Weinberg-Joos formalism reported in [16], where the calculation has been carried out while treating the states under consideration as eight-component spinors. The only difference concerns the value of the gyromagnetic ratio which in the present work comes out fixed to the inverse of the spin, $g=\frac{2}{3}$, and in accord with Belinfante's conjecture, while in [16], a method exclusively based on the Poincaré covariant spin-projector alone,

g had remained unspecified according to,

$$[Q_E^0(\lambda)]_{TS} = e, \tag{71}$$

$$[Q_M^1(\lambda)]_{TS} = g\left(\frac{e}{2m}\right)\langle M_{12}\rangle, \tag{72}$$

$$[Q_E^2(\lambda)]_{TS} = -(1-g)\left(\frac{e}{m^2}\right)\langle \mathcal{A}\rangle, \tag{73}$$

$$[Q_M^3(\lambda)]_{TS} = -3g\left(\frac{e}{2m^3}\right)\langle \mathcal{B}\rangle. \tag{74}$$

We conclude that the anti-symmetric tensor-spinor provides a description of particles of spin-$\frac{3}{2}$ transforming as $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ that is equivalent to the Weinberg-Joos formalism. The great advantage of the tensor basis employed here that it allows to comfortably carry out more complicated calculations such a Compton scattering, the subject of the next section.

4.5 Compton scattering off spin-$\frac{3}{2}$ particles in $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$

The tree-level Compton scattering amplitude contains contributions from the three different channels displayed in the Figs. 4, 5, 6. In denoting by p and p' in turn the four-momenta of the incident and scattered single spin-$\frac{3}{2}$ target particles, while by q and q' the four-momenta of the incident and scattered photons, respectively, the amplitude takes the following form [26],

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3. \tag{75}$$

In the following we shall evaluate the process off positive parity states, in which case one has,

$$
\begin{aligned}
i\mathcal{M}_1 = e^2 \left[\overline{w}_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p}',\frac{3}{2},\lambda'\right)\right]^A & \qquad [\mathcal{U}_{\mu\nu}(p',Q,p)]_{AB} \\
\left[w_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^B & \qquad [\epsilon^\mu(\mathbf{q}',\ell')]^*\,\epsilon^\nu(\mathbf{q},\ell),
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
i\mathcal{M}_2 = e^2 \left[\overline{w}_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p}',\frac{3}{2},\lambda'\right)\right]^A & \qquad [\mathcal{U}_{\nu\mu}(p',R,p)]_{AB} \\
\left[w_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^B & \qquad [\epsilon^\mu(\mathbf{q}',\ell')]^*\epsilon^\nu(\mathbf{q},\ell),
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
-i\mathcal{M}_3 = e^2 \left[\overline{w}_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p}',\frac{3}{2},\lambda'\right)\right]^A & \qquad [\mathcal{X}_{\mu\nu}]_{AB} \\
\left[w_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^B & \qquad [\epsilon^\mu(\mathbf{q}',\ell')]^*\,\epsilon^\nu(\mathbf{q},\ell) \\
,
\end{aligned}
\tag{78}
$$

with $Q = p + q' = p' + q'$ and $R = p' - q = p - q'$. Furthermore, we define the short-hands,

$$
\begin{aligned}
[\mathcal{U}_{\mu\nu}(p',Q,p)]_{AB} &= \left[\mathcal{V}_\mu^{\left(\frac{3}{2},0\right)}(p',Q)\right]_{AC} \left[S^{\left(\frac{3}{2},0\right)}(Q)\right]^{CD} \\
&\quad \left[\mathcal{V}_\nu^{\left(\frac{3}{2},0\right)}(Q,p)\right]_{DB}, \qquad (79)\\
[\mathcal{X}_{\mu\nu}]_{AB} &= \left[\mathcal{C}_{\mu\nu}^{\left(\frac{3}{2},0\right)} + \mathcal{C}_{\nu\mu}^{\left(\frac{3}{2},0\right)}\right]_{AB}, \qquad (80)
\end{aligned}
$$

with $\mathcal{V}_\mu^{\left(\frac{3}{2},0\right)}$ and $C_{\mu\nu}^{\left(\frac{3}{2},0\right)}$ defined in (62) and (63), respectively, and the indexes A,B being introduced in (56). The gauge invariance of this amplitude is ensured by the Ward-Takahashi identity in (64). The averaged squared

amplitude then reads,

$$\overline{|\mathcal{M}|^2} = \frac{1}{8}\sum_{\lambda,\lambda',\ell,\ell'} \mathcal{M}[\mathcal{M}]^\dagger \tag{81}$$

$$= \frac{1}{8} Tr\left[[\mathcal{M}_{\mu\nu}(p',Q,R,p)]_{AB}[\mathcal{M}^{\nu\mu}(p,R,Q,p')]^{AB}\right], \tag{82}$$

and the corresponding expression contains the projector on positive parity states, $\mathbf{P}_+^{\left(\frac{3}{2},0\right)}(\mathbf{p}')$, as

$$\begin{aligned}
[\mathcal{M}_{\mu\nu}(p',Q,R,p)]_{AB} &= e^2[\mathbf{P}_+^{\left(\frac{3}{2},0\right)}(\mathbf{p}')]_A{}^C[\mathbf{U}_{\mu\nu}(p',Q,R,p)]_{CB}, \\
[\mathbf{U}_{\mu\nu}(p',Q,R,p)]_{CB} &= \Big(\mathcal{U}_{\mu\nu}(p',Q,p) + \mathcal{U}_{\nu\mu}(p',R,p) - \mathcal{X}_{\mu\nu}\Big)_{CB}.
\end{aligned} \tag{83}$$

The projector over spin-$\frac{3}{2}$ states of positive parity has same structure as the one given by [27],

$$[\mathbf{P}_+^{\left(\frac{3}{2},0\right)}(\mathbf{p})]_{AB} = \sum_\lambda \left[w_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]_A \left[\overline{w}_+^{\left(\frac{3}{2},0\right)}\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]_B. \tag{84}$$

We furthermore used

$$\sum_\ell \epsilon^\mu(\mathbf{q},\ell)[\epsilon^\nu(\mathbf{q},\ell)]^* = -g^{\mu\nu}. \tag{85}$$

The pure spin-$\frac{3}{2}$ projector, $[\mathbf{P}_+^{\left(\frac{3}{2},0\right)}(\mathbf{p})]_{AB}$ can equivalently be rewritten to

$$\left[\mathbf{P}_+^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]_{AB} = \left[f^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]_A{}^\mu \left(\frac{-\not{p}+m}{2m}\right)\left[\overline{f}^{\left(\frac{3}{2},0\right)}(\mathbf{p})\right]_{B\mu}, \tag{86}$$

with $f^{\left(\frac{3}{2},0\right)}(\mathbf{p})$ from (37). The contractions indicated in (82) are comfortably evaluated with the aid of the FeynCalc symbolic software package amounting to:

$$\overline{|\mathcal{M}|^2} = \frac{1}{162m^6\left(m^2-s\right)^2\left(m^2-u\right)^2}\sum_{k=1}^{7} m^{2k}a_{2k}, \tag{87}$$

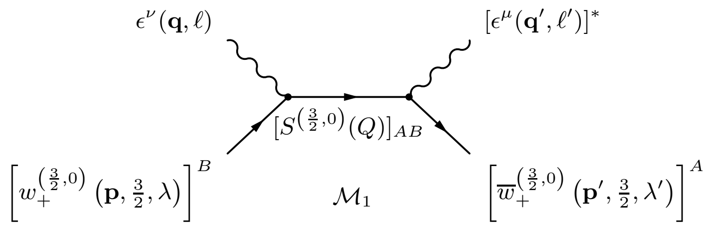

Fig. 4: Diagram for the direct-scattering contribution (76) to the Compton scattering amplitude (75).

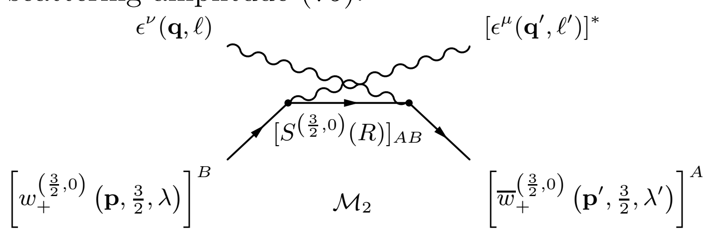

Fig. 5: Diagram for the exchange-scattering contribution (77) to the Compton scattering amplitude (75).

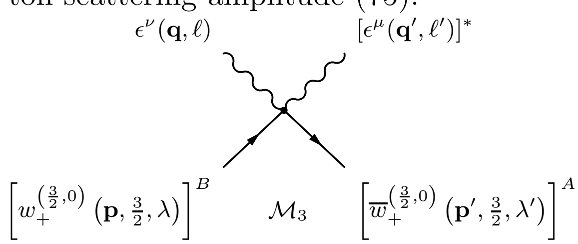

Fig. 6: Diagram for the point-scattering contribution (78) to the Compton scattering amplitude (75).

where s, u are the standard Mandelstam variables and we are using the notations

$$\begin{aligned}
a_0 &= 18s^2u^2(s+u)^3, &(88)\\
a_2 &= -9su(s+u)^2(7(s^2+u^2)+8su), &(89)\\
a_4 &= (s+u)(63(s^4+u^4)\\
&+ 348(s^3u+su^3)+578s^2u^2), &(90)\\
a_6 &= -165(s^4+u^4)-588(s^3u+su^3)-574s^2u^2, &(91)\\
a_8 &= 2(s+u)(5(s^2+u^2)-142su), &(92)\\
a_{10} &= 2(105(s^2+u^2)-158su), &(93)\\
a_{12} &= -280(s+u), &(94)\\
a_{14} &= 912. &(95)
\end{aligned}$$

The differential cross section in the laboratory frame is now calculated from the standard formulas,

$$\begin{aligned}
\frac{d\boldsymbol{\sigma}}{d\Omega} &= \left(\frac{1}{8\pi m}\frac{\omega'}{\omega}\right)^2 \overline{|\mathcal{M}|^2}, &(96)\\
\omega' &= \frac{m\omega}{m+(1-\cos\theta)\omega}, &(97)
\end{aligned}$$

where ω and ω' are the energies of the incident and scattered photons respectively, while θ is the scattering angle in the laboratory frame. Furthermore, with

$$\begin{aligned}
s &= m(m+2\omega), &(98)\\
u &= m(m-2\omega'), &(99)
\end{aligned}$$

and after some algebraic manipulations, the final result is cast in the form of an expansion in powers of η^k (with $\eta = \omega/m$) according to,

$$\frac{d\boldsymbol{\sigma}(\eta, x)}{d\Omega} = \frac{r_0^2}{162(\eta(x-1)-1)^5}\sum_{k=0}^{6}\eta^k b_k. \quad (100)$$

Here, $r_0 = e^2/(4\pi m) = \alpha/m$, $x = \cos\theta$, and the expansion coefficients are,

$$
\begin{aligned}
b_0 &= -81(x^2+1), && (101)\\
b_1 &= 243(x-1)(x^2+1), && (102)\\
b_2 &= -(x-1)(243x^3-333x^2+338x-468), && (103)\\
b_3 &= (x-1)^2(81x^3-261x^2+271x-531), && (104)\\
b_4 &= (x-1)^2(90x^3-233x^2+440x-459), && (105)\\
b_5 &= 6(x-1)^3(8x^2-20x+39), && (106)\\
b_6 &= 9(x-1)^3(x^2-5x+8). && (107)
\end{aligned}
$$

In the low energy limit, we recover as expected the correct expression for the differential cross section in the Thompson limit as,

$$
\lim_{\eta\to 0}\frac{d\boldsymbol{\sigma}(\eta,x)}{d\Omega} = \frac{1}{2}r_0^2\left(x^2+1\right), \tag{108}
$$

while in forward direction, the differential cross section takes an energy independent value,

$$
\lim_{x\to 1}\frac{d\boldsymbol{\sigma}(\eta,x)}{d\Omega} = r_0^2, \tag{109}
$$

and in accord with unitarity. In all the other directions however, the differential cross section increases with energy. In the Figure 7 we present a plot of the quantity

$$
d\tilde{\boldsymbol{\sigma}} = \frac{1}{r_0^2}\frac{d\boldsymbol{\sigma}(\eta,x)}{d\Omega} \tag{110}
$$

as a function of the $x = \cos\theta$ variable, at energies of $\eta = 0$ (solid curve), $\eta = 1$ (long dashed curve) and $\eta = 2.5$ (short dashed curve), here we see how the differential cross section approaches the classical limit at low energy (symmetric curve) and raises as the energy grows except in the forward direction.

Integrating over the solid angle we find the total cross section as:

$$
\boldsymbol{\sigma}(\eta) = \sum_{k=0}^{8}\frac{\eta^k c_k\boldsymbol{\sigma}_T}{108\eta^2(2\eta+1)^4} + \sum_{\ell=0}^{4}\frac{\eta^\ell h_\ell\boldsymbol{\sigma}_T\log(2\eta+1)}{216\eta^3}, \tag{111}
$$

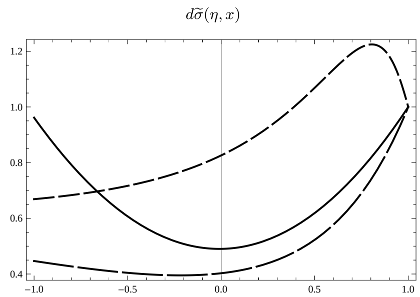

Fig. 7: The differential cross section, $d\widetilde{\boldsymbol{\sigma}}(\eta, x)$, as a function of $x = \cos\theta$. The solid curve represents the classical limit at $\eta = \omega/m = 0$, the long dashed line corresponds to an energy comparable to the mass of the particle, $\eta = 1$, while the short dashed curve corresponds to $\eta = 2.5$. This cross section increases with energy, except in the forward direction, $x = 1$, where it approaches $d\widetilde{\boldsymbol{\sigma}}(\eta, 1) = 1$.

being $\boldsymbol{\sigma}_T = (8/3)\pi r_0^2$ the Thompson cross section and

$$\begin{aligned} c_0 &= 162, \quad c_1 = 1566, && (112)\\ c_2 &= 6217, \quad c_3 = 12796, && (113)\\ c_4 &= 14244, \quad c_5 = 8011, && (114)\\ c_6 &= 1794, \quad c_7 = 126, && (115)\\ c_8 &= 72, \quad h_0 = -162, && (116)\\ h_1 &= -432, \quad h_2 = -277, && (117)\\ h_3 &= -21, \quad h_4 = 27. && (118) \end{aligned}$$

The total cross section (111) has the following limits,

$$\begin{aligned} \lim_{\eta\to 0} \boldsymbol{\sigma}(\eta) &= \boldsymbol{\sigma}_T, && (119)\\ \lim_{\eta\to\infty} \boldsymbol{\sigma}(\eta) &= \infty. && (120) \end{aligned}$$

This behavior is show in Figure 8, where we make a plot of

$$\tilde{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}(\eta)}{\boldsymbol{\sigma}_T}, \qquad (121)$$

here we see the decreasing behavior of the cross section at low energies as well as its growing behavior at high energies.

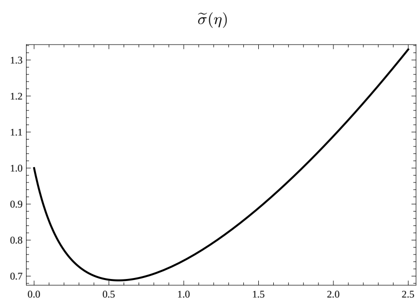

Fig. 8: The total cross section $\widetilde{\sigma}(\eta)$ as a function of $\eta = \omega/m$. In the low energy limit the Thompson limit, $\widetilde{\sigma}(0) = 1$, is recovered, otherwise the cross section grows with the energy increase.

5 Conclusions

The first achievement of this work is the representation in the equations (33)-(34) of the spin-$\frac{3}{2}$ degrees of freedom spanning the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ carrier space of the Lorentz algebra as anti-symmetric second rank Lorentz tensor-spinors. This representation is equipped by separate Lorentz and Dirac indexes and provides a comfortable tool in calculations of scattering cross sections by means of the symbolic software package FeynCalc. A similar experience has been made in [17] regrading spin-1 transforming as $(1,0) \oplus (0,1)$, where the calculation of Compton scattering could not be tackled in terms of six-dimensional spinors and was instead easily executed in the anti-symmetric tensor-basis. We here specifically worked out the Compton scattering off $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ and reported in eq. (109) on finite differential cross section in forward direction in the ultra relativistic regime and in accord with unitarity. Before we had calculated in the eqs. (66)–(69) the multipole moments of a particle transforming according to the representation space under investigation and found that they reproduced results earlier predicted by the Weinberg-Joos theory, the gyromagnetic ratio coming out fixed to the inverse of the spin, i.e. $g = \frac{1}{j} = \frac{2}{3}$ and in accord with Belinfante's conjecture. It is this very same value that gives rise to unitarity in the process of Compton scattering in forward direction and at variance with the $g = 2$ value required by spin-$\frac{3}{2}$ transforming in the four-vector spinor. Our observation suggests that spin-$\frac{3}{2}$ particles transforming within the two-spin valued four-vector spinor, on the one side, and as the single spin valued $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$, on the other side, behave as distinct species. Within the framework of the

second order approach elaborated here, we provided arguments in favor of the causal propagation of a pure spin-$\frac{3}{2}$ interacting with an electromagnetic field. We furthermore explained how our scheme allows for an extension toward any spin, be it bosonic or fermionic, in retaining the quadratic wave equations and associated Lagrangians. Our conclusion is that $(j,0)\oplus(0,j)$ representation spaces preserve their individuality upon embeddings into reducible Lorentz-tensors. We have checked that also all the other irreducible sectors of the antisymmetric tensor-spinor, namely, the single spin-$\frac{1}{2}$ Dirac sector, as well as the double spin valued $\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right)$ Rarita-Schwinger, are characterized before and after embeddings by same sets of multiple moments and Compton scattering cross sections. The embeddings under discussion bring the advantage of separate Lorentz and Dirac indexes which significantly speeds up the tensor calculus relatively to the matrix calculus.

Appendix: The explicit spin- $\frac{3}{2}$ degrees of freedom contained within the anti-symmetric tensor spinor

As long as according to (2) the anti-symmetric tensor spinor represents itself as the direct sum of $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ (8 degrees of freedom) with a Dirac spinor, $\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$ (4 degrees of freedom), and $\left(\frac{1}{2},1\right)\oplus\left(1,\frac{1}{2}\right)$ (12 degrees of freedom), the number of degrees of freedom is this space is 24. The set of the first 16 four-vector spinor degrees of freedom has been constructed in [14], with those corresponding to spin-$\frac{3}{2}$ in $\left(\frac{1}{2},1\right)\oplus\left(1,\frac{1}{2}\right)$ being

$$
\begin{aligned}
\mathcal{U}^{\alpha}_{\pm}\left(\mathbf{p},\frac{3}{2},\frac{3}{2}\right) &= \eta^{\alpha}(\mathbf{p},1,1)u_{\mp}\left(\mathbf{p},\frac{1}{2}\right),\\
\mathcal{U}^{\alpha}_{\pm}\left(\mathbf{p},\frac{3}{2},\frac{1}{2}\right) &= \sqrt{\frac{1}{3}}\eta^{\alpha}(\mathbf{p},1,1)u_{\mp}\left(\mathbf{p},-\frac{1}{2}\right)\\
&+ \sqrt{\frac{2}{3}}\eta^{\alpha}(\mathbf{p},1,0)u_{\mp}\left(\mathbf{p},\frac{1}{2}\right),\\
\mathcal{U}^{\alpha}_{\pm}\left(\mathbf{p},\frac{3}{2},-\frac{1}{2}\right) &= \sqrt{\frac{1}{3}}\eta^{\alpha}(\mathbf{p},1,-1)u_{\mp}\left(\mathbf{p},\frac{1}{2}\right)\\
&+ \sqrt{\frac{2}{3}}\eta^{\alpha}(\mathbf{p},1,0)u_{\mp}\left(\mathbf{p},-\frac{1}{2}\right),\\
\mathcal{U}^{\alpha}_{\pm}\left(\mathbf{p},\frac{3}{2},-\frac{3}{2}\right) &= \eta^{\alpha}(\mathbf{p},1,-1)u_{\mp}\left(\mathbf{p},-\frac{1}{2}\right).
\end{aligned}
\tag{122}
$$

Here $u_{\mp}(\mathbf{p},\lambda)$ are negative and positive energy Dirac spinors, coupled to the following spin-1 four-vectors, $\eta^{\alpha}(\mathbf{p},\lambda)$ spanning $\left(\frac{1}{2},\frac{1}{2}\right)$, given (modulo notational differences) by [24]

$$
\eta^{\alpha}(\mathbf{p},1,1) = \frac{1}{\sqrt{2}m(m+p_0)}\begin{pmatrix} -(m+p_0)(p_1+ip_2)\\ -m^2-p_0m-p_1^2-ip_1p_2\\ -i(p_2^2-ip_1p_2+m(m+p_0))\\ -(p_1+ip_2)p_3 \end{pmatrix}, \tag{123}
$$

$$
\eta^{\alpha}(\mathbf{p},1,0) = \frac{1}{m(m+p_0)}\begin{pmatrix} (m+p_0)p_3\\ p_1p_3\\ p_2p_3\\ p_3^2+m(m+p_0) \end{pmatrix}, \tag{124}
$$

$$
\eta^{\alpha}(\mathbf{p},1,-1) = \frac{1}{\sqrt{2}m(m+p_0)}\begin{pmatrix} (m+p_0)(p_1-ip_2)\\ m^2+p_0m+p_1^2-ip_1p_2\\ -i(p_2^2+ip_1p_2+m(m+p_0)).\\ (p_1-ip_2)p_3 \end{pmatrix} \tag{125}
$$

There is one more four-vector residing within the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space which is of spin-0 and reads [24],

$$
\text{spin}-0\in\left(\frac{1}{2},\frac{1}{2}\right):\quad \eta^{\alpha}(\mathbf{p},0,0)=\frac{p^{\alpha}}{m}. \tag{126}
$$

The coupling of the latter vector to $\mathcal{U}^\alpha_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)$ in the equations (122) from above provides a new independent orthogonal complete set of spin-$\frac{3}{2}$ states according to,

$$\left[U_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]} = \frac{1}{2}\Big(\eta^\alpha(\mathbf{p},0,0)\mathcal{U}^\beta_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right) - \eta^\beta(\mathbf{p},0,0)\mathcal{U}^\alpha_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\Big) \tag{127}$$

$$= \frac{1}{2}\Big(\frac{p^\alpha}{m}\mathcal{U}^\beta_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right) - \frac{p^\beta}{m}\mathcal{U}^\alpha_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\Big). \tag{128}$$

These are the tensor-spinors which, according to (34), find themselves at the root of the degrees of freedom of the $\left(\frac{3}{2},0\right)\oplus\left(0,\frac{3}{2}\right)$ representation space. Finally, the conjugate states to $\overline{\mathcal{U}}^\alpha_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)$, and $\left[\overline{U}_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]}$ are defined as

$$\overline{\mathcal{U}}^\alpha_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right) = \left[\gamma_0\mathcal{U}_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{\dagger\alpha}, \tag{129}$$

$$\left[\overline{U}_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{[\alpha\beta]} = \left[\gamma_0 U_\pm\left(\mathbf{p},\frac{3}{2},\lambda\right)\right]^{\dagger[\alpha\beta]}. \tag{130}$$

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